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# On the availability of quandle theory to classifying links up to link-homotopy

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## 1 Introduction

The notion of link-homotopy, introduced by Milnor [12], gives rise to an equivalence relation on oriented and ordered links in  $S^3$ . Two links are said to be link-homotopic if they are related to each other by a finite sequence of ambient isotopies and self-crossing changes, keeping the orientation and ordering. Here, a self-crossing change is a homotopy for a single component of a link depicted in Figure 1, supported in a small ball whose intersection with the component consists of two segments. The classification problem of links up to link-homotopy is already solved by Habegger and Lin [5] completely. They gave an algorithm which determines whether given links are link-homotopic or not. On the other hand, a table consisting of all representatives of link-homotopy classes is still not known other than partial ones given by Milnor [12, 13] for links with 3 or fewer components and by Levine [10] for links with 4 components. The comparison algorithm never gives us a complete table. To obtain such a table, we should require link-homotopy invariants. Indeed, both of Milnor and Levine utilized numerical invariants to obtain the tables.

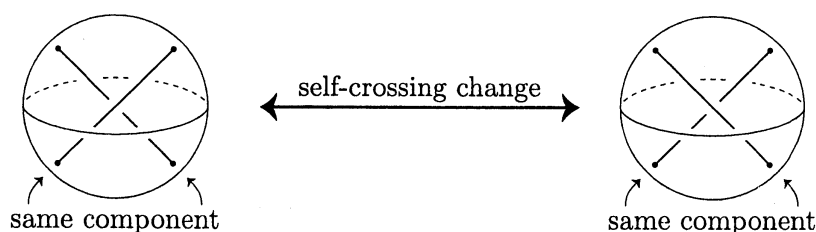


Figure 1:

A quandle, introduced by Joyce [9], is an algebraic system consisting of a set together with a binary operation whose definition is strongly motivated in knot theory. Joyce

defined the knot quandle of a link so that knot quandles are isomorphic if associated links are ambient isotopic to each other. Furthermore, Carter et al. [1, 2] introduced homology of a quandle and showed that we have the fundamental classes in the second quandle homology group of a knot quandle being invariant under ambient isotopy. Each homomorphism from a knot quandle to a quandle induces a homomorphism from the second quandle homology group of the knot quandle to that of the quandle, as usual. Thus the multi-set consisting of the values obtained by evaluating the images of the fundamental classes by all these induced homomorphisms with a 2-cocycle of the quandle is invariant under ambient isotopy. We call this multi-set a quandle cocycle invariant.

Although knot quandles are not invariant under link-homotopy, Hughes [6] showed that their quotients, called reduced knot quandles, are invariant under link-homotopy. The author [7] showed that, if we modify the definition of quandle homology slightly, then we still have the fundamental classes in the second quandle homology group of a reduced knot quandle being invariant under link-homotopy. We thus have a quandle cocycle invariant which is invariant under link-homotopy.

The latent ability of quandle cocycle invariants for classifying links up to link-homotopy essentially depends on the power of reduced knot quandles and their fundamental classes for classifying links. The author conjectures that a pair of a reduced knot quandle and its fundamental classes is a complete link-homotopy invariant (Conjecture 4.1). In this paper, we show that this conjecture is true under some conditions (Theorem 4.2).

Throughout this paper, links are assumed to be oriented, ordered and in  $S^3$ .

## Acknowledgements

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## 2 Quandle cocycle invariant for ambient isotopy

In this section, we review a quandle cocycle invariant for ambient isotopy briefly.

A *quandle* is a non-empty set  $X$  equipped with a binary operation  $*$  :  $X \times X \rightarrow X$  satisfying the following three axioms:

- (Q1) For each  $x \in X$ ,  $x * x = x$ .
- (Q2) For each  $x \in X$ , a map  $*x : X \rightarrow X$  ( $w \mapsto w * x$ ) is bijective.
- (Q3) For each  $x, y, z \in X$ ,  $(x * y) * z = (x * z) * (y * z)$ .

The notion of a homomorphism between quandles is appropriately defined. We will write the image  $(*y)^\varepsilon(x)$  as  $x *^\varepsilon y$  for any  $x, y \in X$  and  $\varepsilon \in \{\pm 1\}$ .

Associated with a link  $L$ , we have a quandle as follows. Let  $N$  be a subspace of  $\mathbb{C}$  which is the union of the closed unit disk  $D$  and a segment  $\{z \in \mathbb{C} \mid 1 \leq z \leq 5\}$ . Assume that  $D$  is oriented counterclockwise. A *noose* of  $L$  is a continuous map  $\nu : N \rightarrow S^3$  satisfying the following conditions:

- The map  $\nu$  sends  $5 \in N$  to a fixed base point  $p \in S^3 \setminus L$ .
- The restriction map  $\nu|_D : D \rightarrow S^3$  is an embedding.
- The link  $L$  intersects with  $\text{Im } \nu$  transversally only at  $\nu(0)$ .
- The intersection number between  $L$  and  $\text{Im } \nu|_D$  is  $+1$ .

The left-hand side of Figure 2 depicts an image of a noose  $\nu$ . We define a product  $*$  of two nooses  $\mu$  and  $\nu$  by

$$(\mu * \nu)(z) = \begin{cases} \mu(z) & \text{if } |z| \leq 1, \\ \mu(4z - 3) & \text{if } 1 \leq z \leq 2, \\ \nu(13 - 4z) & \text{if } 2 \leq z \leq 3, \\ \nu(\exp(2(z - 3)\pi i)) & \text{if } 3 \leq z \leq 4, \\ \nu(4z - 15) & \text{if } 4 \leq z \leq 5. \end{cases}$$

The right-hand side of Figure 2 shows what happens if we take this product. Let  $Q(L)$  be the set consisting of all homotopy classes of nooses of  $L$ . The product  $*$  of nooses is obviously well-defined on  $Q(L)$  and satisfies the axioms of a quandle. We call this quandle  $Q(L)$  with  $*$  the *knot quandle* of  $L$ . By definition, a knot quandle is obviously invariant under ambient isotopy. Thus the set consisting of all homomorphisms from a knot quandle to a quandle gives rise to an invariant of links. Especially, the cardinality of the set is a numerical invariant.

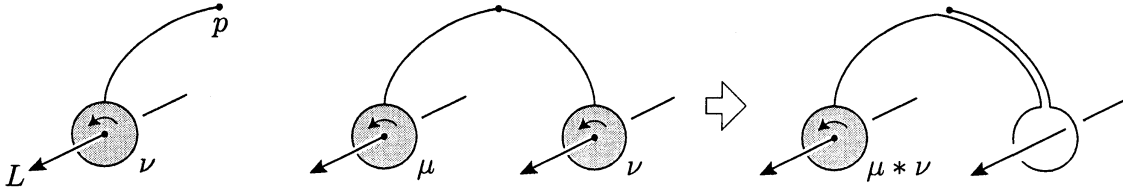


Figure 2:

For a quandle  $X$ , consider the free abelian group  $C_n^R(X)$  generated by all  $n$ -tuples  $(x_1, x_2, \dots, x_n) \in X^n$  for each  $n \geq 1$ . We let  $C_0^R(X) = \mathbb{Z}$ . Define a map  $\partial_n : C_n^R(X) \rightarrow$

$C_{n-1}^R(X)$  by

$$\begin{aligned} \partial_n(x_1, x_2, \dots, x_n) = & \sum_{i=2}^n (-1)^i \{ (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \\ & - (x_1 * x_i, \dots, x_{i-1} * x_i, x_{i+1}, \dots, x_n) \} \end{aligned}$$

for  $n \geq 2$ , and  $\partial_1 = 0$ . Then we have  $\partial_{n-1} \circ \partial_n = 0$ . Thus  $(C_n^R(X), \partial_n)$  is a chain complex. Let  $C_n^D(X)$  be a subgroup of  $C_n^R(X)$  generated by  $n$ -tuples  $(x_1, x_2, \dots, x_n) \in X^n$  with  $x_i = x_{i+1}$  for some  $i$  if  $n \geq 2$ , and let  $C_n^D(X) = 0$  otherwise. It is routine to check that  $\partial_n(C_n^D(X)) \subset C_{n-1}^D(X)$ . Therefore, putting  $C_n^Q(X) = C_n^R(X)/C_n^D(X)$ , we have a chain complex  $(C_n^Q(X), \partial_n)$ . Let  $G$  be an abelian group. The  $n$ -th *quandle homology group*  $H_n^Q(X; G)$  with coefficients in  $G$  is the  $n$ -th homology group of the chain complex  $(C_n^Q(X) \otimes G, \partial_n \otimes \text{id})$ . The  $n$ -th *quandle cohomology group*  $H_Q^n(X; G)$  with coefficients in  $G$  is the  $n$ -th cohomology group of the cochain complex  $(\text{Hom}(C_n^Q(X), G), \text{Hom}(\partial_n, \text{id}))$ . We will use the symbol  $[\cdot]$  to denote a class of quandle homology or cohomology.

Let  $L$  be a link and  $D$  its diagram. To arcs  $\alpha, \beta, \dots$  of  $D$ , we assign elements  $a, b, \dots$  of the knot quandle  $Q(L)$  respectively in the same manner as Wirtinger generators. For the  $i$ -th component of  $L$ , consider an element  $W_i = \sum \varepsilon \cdot (a, b) \in C_2^Q(Q(L))$ , where the sum runs over the crossings of  $D$  which consist of under arcs  $\alpha$  and  $\gamma$  belonging to the  $i$ -th component and an over arc  $\beta$  (see Figure 3), and  $\varepsilon$  is 1 or  $-1$  depending on whether the crossing is positive or negative respectively. Then, by construction,  $W_i$  is a 2-cycle. Suppose  $D'$  is a diagram of  $L$  obtained from  $D$  by a single Reidemeister move and  $W'_i \in C_2^Q(Q(L))$  the 2-cycle derived from  $D'$ . The axioms of a quandle ensure that the difference  $W'_i - W_i$  is in the second boundary group  $B_2^Q(Q(L))$  (see [1, 2]). Thus the class  $[W_i] \in H_2^Q(Q(L))$  does not depend on the choice of  $D$ , i.e., it is invariant under ambient isotopy. We call this class the *fundamental class* of the knot quandle  $Q(L)$  derived from the  $i$ -th component, and denote it by  $[K_i] \in H_2^Q(Q(L))$ .

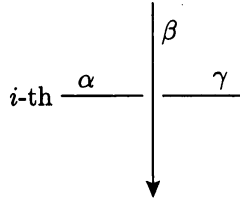


Figure 3:

Let  $X$  be a quandle,  $G$  an abelian group and  $\theta \in \text{Hom}(C_2^Q(X), G)$  a 2-cocycle. For an  $n$ -component link  $L$ , consider the multi-set consisting of  $n$ -tuples

$$(\langle [\theta] | f | [K_1] \rangle, \langle [\theta] | f | [K_2] \rangle, \dots, \langle [\theta] | f | [K_n] \rangle) \in G^n$$

derived from all homomorphisms  $f : Q(L) \rightarrow X$ , where  $\langle [\theta] | f | [K_i] \rangle \in G$  denotes the value obtained by evaluating the image of  $[K_i] \in H_2^Q(Q(L))$  by the homomorphism  $H_2^Q(Q(L)) \rightarrow H_2^Q(X)$  induced from  $f$  with  $[\theta] \in H_2^Q(X; G)$ . This multi-set, introduced by Carter et al. [2], is obviously invariant under ambient isotopy and is called a *quandle cocycle invariant*.

### 3 Quandle cocycle invariant for link-homotopy

It is known by Joyce [9] and independently by Matveev [11] that knot quandles of knots (1-component links) are isomorphic if and only if associated knots are weak equivalent, i.e., there is a homeomorphism of  $S^3$  sending an associated knot to the other. On the other hand, every knots are trivial up to link-homotopy. Therefore, knot quandles are not invariant under link-homotopy. It means that quandle cocycle invariants are not invariant under link-homotopy in general. However, in this section, we review a certain quotient of a knot quandle, called a reduced knot quandle, is invariant under link-homotopy. Further, modifying the definition of quandle homology slightly, we have a quandle cocycle invariant being invariant under link-homotopy.

For homotopy classes of nooses of a link  $L$ , consider the moves depicted in Figure 4. We let  $RQ(L)$  be the quotient of the set  $Q(L)$  consisting of all homotopy classes of nooses of  $L$  by the moves. Then the product  $*$  of nooses is still well-defined on  $RQ(L)$  and satisfies the axioms of a quandle. We call this quandle  $RQ(L)$  with  $*$  the *reduced knot quandle* of  $L$ . It is known by Hughes [6] that reduced knot quandles are isomorphic if associated links are link-homotopic<sup>1</sup>.

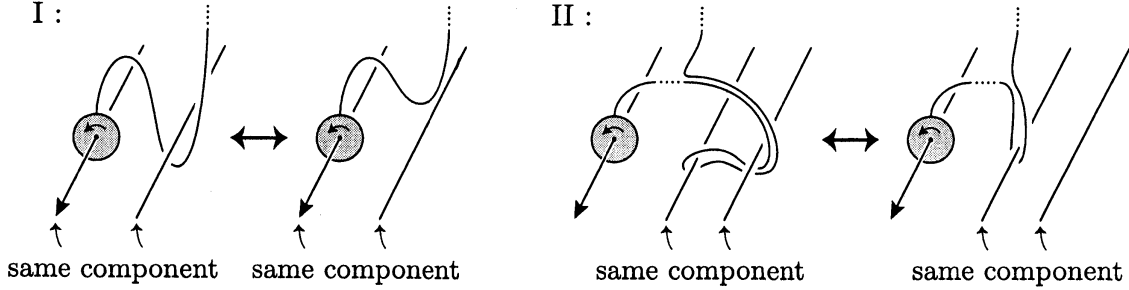


Figure 4:

To discuss an algebraic property of a reduced knot quandle, we first review the following notions. For a quandle  $X$ , an *automorphism group*  $\text{Aut}(X)$  is defined to be the group

<sup>1</sup>This definition of a reduced knot quandle is given by the author. In his paper [6], Hughes defined a reduced knot quandle in an algebraic way and a more complicated geometric way.

consisting of all automorphisms of  $X$ . The axiom (Q3) of a quandle says that the bijection  $*x : X \rightarrow X$  is an automorphism of  $X$  for each  $x \in X$ . An *inner automorphism group*  $\text{Inn}(X)$  of  $X$  is the subgroup of  $\text{Aut}(X)$  generated by the automorphisms  $*x : X \rightarrow X$ . We call an element of the inner automorphism group an *inner automorphism*.

Nooses  $\mu$  and  $\nu$  of a link  $L$  intersect with the same component if and only if there is an inner automorphism of the knot quandle  $Q(L)$  sending the homotopy class of  $\mu$  to that of  $\nu$ . Thus a type I move depicted in Figure 4 is algebraically described as the following relation in  $Q(L)$ :

(QT) For each  $a \in Q(L)$  and  $\varphi \in \text{Inn}(Q(L))$ ,  $a * \varphi(a) = a$ .

Further a type II move depicted in Figure 4 is described as the relation  $a * (b * \varphi(b)) = a * b$  for each  $a, b \in Q(L)$  and  $\varphi \in \text{Inn}(Q(L))$ . Since this relation is a consequence of the relation (QT), the reduced knot quandle  $RQ(L)$  is algebraically described as the quotient of  $Q(L)$  by the relation (QT).

We call a quandle  $X$  to be *quasi-trivial* [7] if  $X$  satisfies the condition  $x * \varphi(x) = x$  for each  $x \in X$  and  $\varphi \in \text{Inn}(X)$ . A reduced knot quandle is of course quasi-trivial. We remark that, for a quandle  $X$  which is not quasi-trivial, there are no homomorphisms other than trivial ones from a reduced knot quandle to  $X$ . Since a reduced knot quandle is invariant under link-homotopy, the set consisting of all homomorphisms from a reduced knot quandle to a (quasi-trivial) quandle gives rise to an link-homotopy invariant. In particular, the cardinality of the set is a numerical invariant.

Let  $L$  be an  $n$ -component link and  $D$  its diagram. For the reduced knot quandle  $RQ(L)$ , we of course have a 2-cycle  $W_i \in C_2^Q(RQ(L))$  derived from  $D$  in the same manner as provided in the previous section. However, if we let  $D''$  be a diagram obtained from  $D$  by a self-crossing change at a crossing of the  $i$ -th component, then the difference  $W_i'' - W_i$  is  $\pm((a, \varphi(a)) + (\varphi(a), a))$  with some  $a \in RQ(L)$  and  $\varphi \in \text{Inn}(RQ(L))$ . This difference is not in the second boundary group  $B_2^Q(RQ(L))$  in general. Therefore, we do not have fundamental classes in  $H_2^Q(RQ(L))$  being invariant under link-homotopy. To solve this problem, we consider to modify the definition of quandle homology as follows.

Suppose  $X$  is a quasi-trivial quandle. Let  $C_n^{D,qt}(X)$  be a subgroup of  $C_n^R(X)$  which is generated by  $n$ -tuples  $(x_1, x_2, \dots, x_n) \in X^n$  with  $x_i = x_{i+1}$  for some  $i$  and elements  $(x_1, \varphi(x_1), x_3, \dots, x_n) + (\varphi(x_1), x_1, x_3, \dots, x_n) \in C_n^R(X)$  for some  $\varphi \in \text{Inn}(X)$  if  $n \geq 2$ . We let  $C_n^{D,qt}(X) = 0$  in the cases  $n = 0, 1$ . By the assumption that  $X$  is quasi-trivial,  $\partial_n(C_n^{D,qt}(X)) \subset C_{n-1}^{D,qt}(X)$ . Therefore, putting  $C_n^{Q,qt}(X) = C_n^R(X) / C_n^{D,qt}(X)$ , we have a chain complex  $(C_n^{Q,qt}(X), \partial_n)$ . For an abelian group  $G$ , let  $H_n^{Q,qt}(X; G)$  denote the  $n$ -th homology group of the chain complex  $(C_n^{Q,qt}(X) \otimes G, \partial_n \otimes \text{id})$ , and  $H_{Q,qt}^n(X; G)$  the  $n$ -th cohomology group of the cochain complex  $(\text{Hom}(C_n^{Q,qt}(X), G), \text{Hom}(\partial_n, \text{id}))$ . We will use the symbol  $[\cdot]^{qt}$  to denote a class of these modified quandle homology or cohomology.

Let  $L$ ,  $D$  and  $D''$  be the same as above. Then we obviously have 2-cycles  $W_i$  and  $W_i''$  in  $C_2^{Q,qt}(RQ(L))$  derived from  $D$  and  $D''$  respectively. Remark that the difference  $W_i'' - W_i$  is equal to zero in  $C_2^{Q,qt}(RQ(L))$  because  $\pm((a, \varphi(a)) + (\varphi(a), a))$  is an element of  $C_2^{D,qt}(RQ(L))$ . Therefore, the homology class  $[W_i]^{qt} \in H_2^{Q,qt}(RQ(L))$  is invariant under link-homotopy. We call this homology class the *fundamental class* of the reduce knot quandle  $RQ(L)$  derived from the  $i$ -th component, and denote it by  $[K_i]^{qt} \in H_2^{Q,qt}(RQ(L))$ .

Let  $X$  be a quasi-trivial quandle,  $G$  an abelian group and  $\theta \in \text{Hom}(C_2^{Q,qt}(X), G)$  a 2-cocycle. Consider the multi-set consisting of  $n$ -tuples

$$(\langle [\theta]^{qt} | f | [K_1]^{qt} \rangle, \langle [\theta]^{qt} | f | [K_2]^{qt} \rangle, \dots, \langle [\theta]^{qt} | f | [K_n]^{qt} \rangle) \in G^n$$

derived from all homomorphisms  $f : RQ(L) \rightarrow X$ . This multi-set, still called a quandle cocycle invariant, is of course invariant under link-homotopy. Using a certain quandle cocycle invariant, we can show a famous fact that the Borromean rings is not trivial up to link-homotopy [7], for example.

**Remark 3.1.** For a quandle  $X$ , let  $F(X)$  be the free group generated by all elements of  $X$  and  $N(X)$  the subgroup of  $F(X)$  normally generated by all elements in the form  $y^{-1}xy(x*y)^{-1}$  with some  $x, y \in X$ . We call the quotient group  $F(X)/N(X)$  the *associated group* of  $X$  and denote it by  $\text{As}(X)$ . Since  $w*(x*y) = ((w*^{-1}y)*x)*y$  for any  $w, x, y \in X$ , we have a homomorphism  $\text{As}(X) \rightarrow \text{Inn}(X)$  sending  $x$  to  $*x$  ( $x \in X$ ). Thus  $\text{As}(X)$  acts on  $X$  from the right through this homomorphism. We will write the image of  $x \in X$  by the right action of  $g \in \text{As}(X)$  as  $x \triangleleft g$ .

For a link  $L$ , it is known that the associated group  $\text{As}(Q(L))$  of the knot quandle  $Q(L)$  is isomorphic to the knot group  $G(L)$  of  $L$  (see [4, 9] for example). An isomorphism  $\text{As}(Q(L)) \rightarrow G(L)$  is given by restricting each noose of  $L$  to the union of  $\partial D$  and the segment  $\{z \in \mathbb{C} \mid 1 \leq z \leq 5\}$  (this is a positive meridian of  $L$ , by definition). Therefore, as Hughes mentioned in [6], the associated group  $\text{As}(RQ(L))$  of the reduced knot quandle  $RQ(L)$  is isomorphic to the reduced knot group  $RG(L)$ . Here,  $RG(L)$  is the quotient group of  $G(L)$  obtained by adding relations which say each positive meridian commutes with all of its conjugates [12].

## 4 Latent ability of quandle cocycle invariants

We remark that the latent ability of quandle cocycle invariants to classifying links up to link-homotopy depends on the abilities of reduced knot quandles, fundamental classes, a choice of a target quandle, and a choice of a 2-cocycle. Especially, for the abilities of reduced knot quandles and their fundamental classes, we have the following conjecture:



**Conjecture 4.1.** *Suppose  $L$  and  $L'$  are  $n$ -component links. We let  $[K_i]^{qt} \in H_2^{Q,qt}(RQ(L))$  and  $[K'_i]^{qt} \in H_2^{Q,qt}(RQ(L'))$  be fundamental classes of  $RQ(L)$  and  $RQ(L')$  respectively. Then  $L$  and  $L'$  are link-homotopic to each other if and only if there is an isomorphism  $f : RQ(L) \rightarrow RQ(L')$  such that  $f_\#([K_i]^{qt}) = [K'_i]^{qt}$  for all  $i$  ( $1 \leq i \leq n$ ), where  $f_\#$  denotes the isomorphism  $H_2^{Q,qt}(RQ(L)) \rightarrow H_2^{Q,qt}(RQ(L'))$  induced from  $f$ .*

If  $L$  is link-homotopic to  $L'$  then obviously there is an isomorphism  $f : RQ(L) \rightarrow RQ(L')$  satisfying  $f_\#([K_i]^{qt}) = [K'_i]^{qt}$  for all  $i$ . The conjecture thus claims that the inverse is also true. If the conjecture is true, we can completely classify links up to link-homotopy by quandle cocycle invariants.

In this section, we show a theorem (Theorem 4.2) which might be useful for trying the conjecture. To express the precise statement of the theorem, we first prepare the following things.

Let  $L$  be an  $n$ -component link. Choose and fix an element  $a_i \in RQ(L)$  intersecting with the  $i$ -th component for each  $i$  ( $1 \leq i \leq n$ ). It is routine to check that  $a_1, a_2, \dots, a_n$  are generators of  $RQ(L)$  (see [6]). For each  $i$ , select a noose  $\nu_i$  representing  $a_i$  so that distinct nooses only intersect in the base point. We note that this choice is not essentially unique. We let  $\mathcal{D}$  be an oriented 2-disk embedded in  $S^3$  in which each  $\nu_i$  is embedded to be compatible with the orientation of  $\mathcal{D}$ . Cutting open  $S^3$  by  $\mathcal{D}$ , we obtain a string link as depicted in Figure 5. Although the choice of  $\mathcal{D}$  is not unique, Habegger and Lin showed that this string link is unique up to link-homotopy [5]. They further showed that the string link is link-homotopic to a pure braid  $\sigma$  (see Figure 5). We note that the closure of  $\sigma$  is of course link-homotopic to  $L$ .

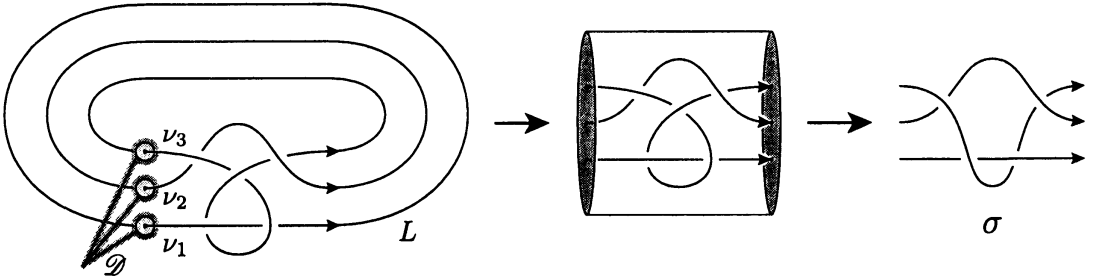


Figure 5:

Suppose  $X$  is a quasi-trivial quandle. Let  $\tilde{C}_n^{D,qt}(X)$  be a subgroup of  $C_n^R(X)$  which is generated by  $n$ -tuples  $(x_1, x_2, \dots, x_n) \in X^n$  with  $x_i = x_{i+1}$  for some  $i$  and  $n$ -tuples  $(x_1, \varphi(x_1), x_3, \dots, x_n) \in X^n$  with some  $\varphi \in \text{Inn}(X)$  for  $n \geq 2$ , and  $\tilde{C}_n^{D,qt}(X) = 0$  for  $n = 0, 1$ . By the assumption that  $X$  is quasi-trivial,  $\partial_n(\tilde{C}_n^{D,qt}(X)) \subset \tilde{C}_{n-1}^{D,qt}(X)$ . Therefore, putting  $\tilde{C}_n^{Q,qt}(X) = C_n^R(X)/\tilde{C}_n^{D,qt}(X)$ , we have a chain complex  $(\tilde{C}_n^{Q,qt}(X), \partial_n)$ . For an

abelian group  $G$ , let  $\tilde{H}_n^{Q,qt}(X; G)$  denote the  $n$ -th homology group of the chain complex  $(\tilde{C}_n^{Q,qt}(X) \otimes G, \partial_n \otimes \text{id})$ , and  $\tilde{H}_{Q,qt}^n(X; G)$  the  $n$ -th cohomology group of the cochain complex  $(\text{Hom}(\tilde{C}_n^{Q,qt}(X), G), \text{Hom}(\partial_n, \text{id}))$ . We will use the symbol  $[\cdot]^{qt}$  again to denote a class of these modified quandle homology or cohomology. We remark that  $C_n^{D,qt}(X)$  is a subgroup of  $\tilde{C}_n^{D,qt}(X)$  and thus  $\tilde{H}_n^{Q,qt}(X; G)$  and  $\tilde{H}_{Q,qt}^n(X; G)$  are quotients of  $H_n^{Q,qt}(X; G)$  and  $H_{Q,qt}^n(X; G)$  respectively. Since any link  $L$  is link-homotopic to a closure of a pure braid, the fundamental classes  $[K_i]^{qt} \in H_2^{Q,qt}(RQ(L))$  are elements in  $\tilde{H}_2^{Q,qt}(RQ(L))$ .

**Theorem 4.2.** *Let  $L$  and  $L'$  be  $n$ -component links. Assume that there is an isomorphism  $f : RQ(L) \rightarrow RQ(L')$  satisfying  $\tilde{f}_\#([K_i]^{qt}) = [K'_i]^{qt}$  for all  $i$  ( $1 \leq i \leq n$ ), where  $\tilde{f}_\#$  denotes the isomorphism  $\tilde{H}_2^{Q,qt}(RQ(L)) \rightarrow \tilde{H}_2^{Q,qt}(RQ(L'))$  induced from  $f$ . If there are pure braids  $\sigma$  and  $\sigma'$  derived from a choice of generators  $a_1, a_2, \dots, a_n \in RQ(L)$  and the generators  $f(a_1), f(a_2), \dots, f(a_n) \in RQ(L')$  respectively such that the pure braids obtained from  $\sigma$  and  $\sigma'$  by removing their  $i$ -th components with some  $i$  are link-homotopic to each other as pure braids, then  $L$  and  $L'$  are link-homotopic to each other.*

The assumption in the last sentence of the theorem is always satisfied for 2-component links. Furthermore, it is routine to check that the assumption is also always satisfied for 3-component links. Therefore, we have the following corollary:

**Corollary 4.3.** *Conjecture 4.1 is true for links with 3 or fewer components.*

To show Theorem 4.2, we first review the following notion. Let  $X$  and  $\tilde{X}$  be (not necessary quasi-trivial) quandles. An epimorphism  $p : \tilde{X} \rightarrow X$  is said to be a *covering* [3] if  $p(\tilde{x}) = p(\tilde{y})$  implies  $\tilde{w} * \tilde{x} = \tilde{w} * \tilde{y}$  for any  $\tilde{w}, \tilde{x}, \tilde{y} \in \tilde{X}$ . In other words, the natural map  $\tilde{X} \rightarrow \text{Inn}(\tilde{X})$  sending  $\tilde{x}$  to  $*\tilde{x}$  factors through  $p$ . This property of a covering enables us to write an element  $\tilde{w} * \tilde{x}$  as  $\tilde{w} * p(\tilde{x})$ .

For each reduced knot quandle, we have its natural coverings as follows. Let  $L$  be an  $n$ -component link and  $\mathcal{D}$  a 2-disk embedded in  $S^3$  derived from a choice of generators  $a_1, a_2, \dots, a_n \in RQ(L)$ . Instead of cutting open  $S^3$  by  $\mathcal{D}$ , we cut open  $S^3$  by a small 2-disk  $\mathcal{D}_i$  in  $\mathcal{D}$  intersecting with  $L$  only at a point of the  $i$ -th component. Then we obtain a  $(1, 1)$ -tangle  $T_i$  as depicted in Figure 6. Consider the reduced knot quandle  $RQ(T_i)$  of  $T_i$  in a similar way. It is easy to see that the projection  $p_i : RQ(T_i) \rightarrow RQ(L)$  derived from the injection  $T_i \rightarrow L$  satisfies the condition for a covering.

Choose a diagram of  $L$  so that the image of  $\mathcal{D}$  is a segment intersecting with each component of  $L$  in order (see the left-hand side of Figure 7). Then removing a small neighborhood of the intersection point between the  $i$ -th component and the image of  $\mathcal{D}$  from the diagram, we have a diagram  $D_i$  of  $T_i$  (see the right-hand side of Figure 7). Let  $\alpha_{ij}$  ( $0 \leq j \leq r_i$ ) denote an arc of  $D_i$  which is a part of the  $i$ -th component in order

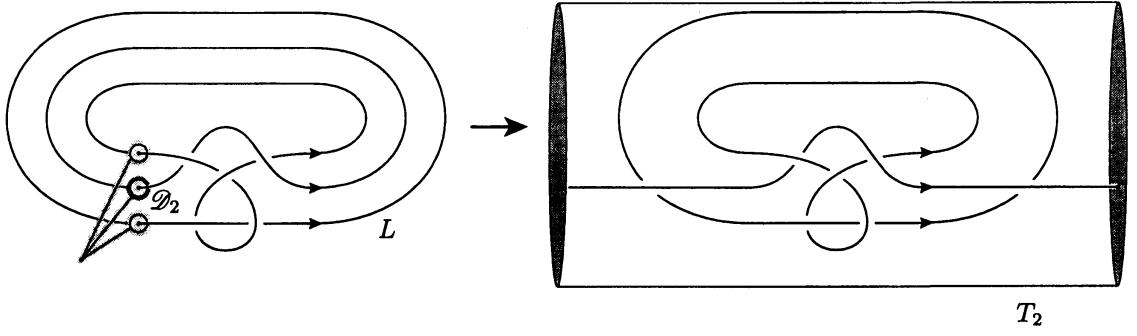


Figure 6:

(see the right-hand side of Figure 7). We assign  $a_{ij} \in RQ(T_i)$  to each  $\alpha_{ij}$  in the same manner as a Wirtinger generator. We note that, although  $p_i(a_{i0})$  and  $p_i(a_{ir_i})$  are the same element,  $a_{i0}$  and  $a_{ir_i}$  are different in general. Let  $\beta_{ij}$  denote the arc separating  $\alpha_{i,j-1}$  and  $\alpha_{ij}$  ( $1 \leq j \leq r_i$ ), and  $b_{ij} \in RQ(T_i)$  the element assigned to  $\beta_{ij}$ . Then we have a relation  $a_{ij} = a_{i,j-1} *^{\varepsilon_{ij}} b_{ij}$  in  $RQ(T_i)$ , where  $\varepsilon_{ij}$  is 1 or  $-1$  depending on whether the crossing consisting of  $\alpha_{i,j-1}$ ,  $\alpha_{ij}$  and  $\beta_{ij}$  is positive or negative respectively.

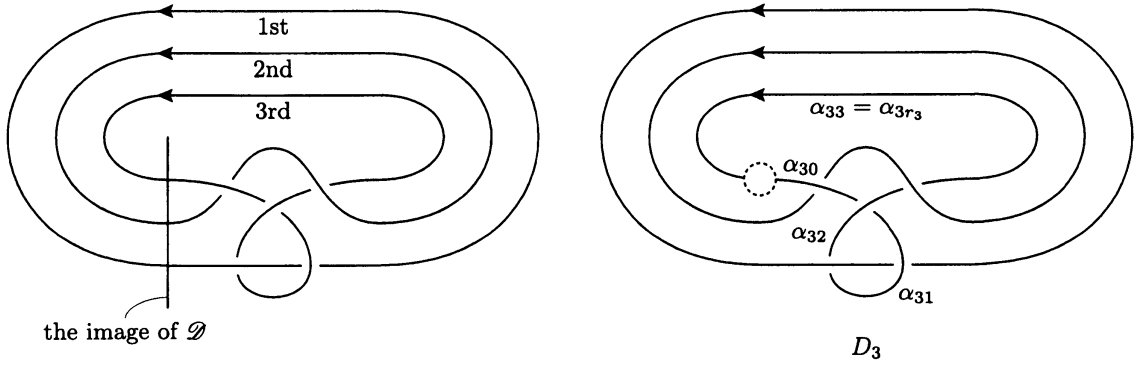


Figure 7:

As mentioned in Remark 3.1, the reduced knot group  $RG(L)$  acts on  $RQ(L)$  from the right. Thus  $RG(L)$  also acts on  $RQ(T_i)$  from the right, because  $p_i : RQ(T_i) \rightarrow RQ(L)$  is a covering. Let  $RG_i(L)$  denote the reduced knot group for the link obtained from  $L$  by removing the  $i$ -th component. Since  $RQ(T_i)$  is quasi-trivial,  $RG_i(L)$  acts on each element of  $RQ(T_i)$  intersecting with the  $i$ -th component from the right through the quotient map  $RG(L) \rightarrow RG_i(L)$ . Therefore, each element of  $RQ(T_i)$  (and also each element of  $RQ(L)$ ) intersecting with the  $i$ -th component can be written as  $a_{i0} \triangleleft u$  with some  $u \in RG_i(L)$ .

Identifying an element of  $RQ(T_i)$  with an element of  $RG_i(L)$ , consider the element

$$l_i = b_{i1}^{\varepsilon_{i1}} b_{i2}^{\varepsilon_{i2}} \cdots b_{ir_i}^{\varepsilon_{ir_i}} \in RG_i(L).$$

Then, by definition, we have  $a_{ir_i} = a_{i0} \triangleleft l_i$ . Milnor [12] showed that  $l_i \in RG_i(L)$  is trivial if and only if the  $i$ -th component of  $L$  is trivial up to link-homotopy. We thus have a cyclic subgroup  $\langle l_i \rangle$  of  $RG_i(L)$ , if the  $i$ -th component is not trivial up to link-homotopy. We note that the order of the cyclic subgroup is not always infinite.

**Lemma 4.4.** *Assume that the  $i$ -th component of  $L$  is not trivial up to link-homotopy. Then we have a 2-cocycle  $\theta_i \in \tilde{Z}_{Q,qt}^2(RQ(L); \langle l_i \rangle)$  which is not in the second coboundary group  $\tilde{B}_{Q,qt}^2(RQ(L); \langle l_i \rangle)$ .*

*Proof.* We first remark that  $p_i$  is not injective, although the restriction of  $p_i$  to the set consisting of all elements not intersecting with the  $i$ -th component is injective. The preimage of  $a_{i0} \triangleleft u \in RQ(L)$  ( $u \in RG_i(L)$ ) by  $p_i$  is the set  $\{a_{i0} \triangleleft l_i^k u \mid k \in \mathbb{Z}_{|\langle l_i \rangle|}\}$ .

Define a left action of  $\langle l_i \rangle$  on  $RQ(T_i)$  by

$$l_i \cdot a = \begin{cases} a_{i0} \triangleleft l_i u & \text{if } a = a_{i0} \triangleleft u \text{ with some } u \in RG(L_i), \\ a & \text{otherwise (i.e., } a \text{ does not intersect with the } i\text{-th component).} \end{cases}$$

Then, associated with a section  $s : RQ(L) \rightarrow RQ(T_i)$  (i.e.,  $s$  is a map satisfying  $p_i \circ s = \text{id}$ ), we have a map  $\theta_i : RQ(L) \times RQ(L) \rightarrow \langle l_i \rangle$  satisfying  $s(a) * s(b) = \theta_i(a, b) \cdot s(a * b)$ . If we set  $\theta(a, b) = 0$  for each  $a \in RQ(L)$  not intersecting with the  $i$ -th component, then  $\theta_i$  is in fact a 2-cocycle and its class does not depend on the choice of a section  $s$  (see the proof of Theorem 4.1 in [8] for more details). By definition, we have  $\langle [\theta_i]^{qt} | \text{id} | [K_j]^{qt} \rangle = l_i^{\delta_{ij}}$ , where  $\delta_{ij}$  denotes the Kronecker delta. Therefore,  $\theta_i$  is not in  $\tilde{B}_{Q,qt}^2(RQ(L); \langle l_i \rangle)$ .  $\square$

**Remark 4.5.** The second last sentence of the above proof says that the fundamental class  $[K_i]^{qt}$  is not trivial if the  $i$ -th component of  $L$  is not trivial up to link-homotopy. Obviously,  $[K_i]^{qt}$  is trivial if the  $i$ -th component of  $L$  is trivial up to link-homotopy.

In the light of Lemma 4.4, we have the following key theorem:

**Theorem 4.6.** *Let  $L$  and  $L'$  be  $n$ -component links. Assume that there is an isomorphism  $f : RQ(L) \rightarrow RQ(L')$  satisfying  $\tilde{f}_\#([K_i]^{qt}) = [K'_i]^{qt}$  for all  $i$  ( $1 \leq i \leq n$ ). Let  $T_i$  and  $T'_i$  denote  $(1, 1)$ -tangles derived from a choice of generators  $a_1, a_2, \dots, a_n \in RQ(L)$  and the generators  $f(a_1), f(a_2), \dots, f(a_n) \in RQ(L')$  respectively. Then we have an isomorphism  $f_i : RQ(T_i) \rightarrow RQ(T'_i)$  sending  $a_{i0}$  to  $a'_{i0}$  and  $a_{ir_i}$  to  $a'_{ir'_i}$ , if the  $i$ -th component of  $L'$  is not trivial up to link-homotopy.*

*Proof.* For each  $k$  ( $1 \leq k \leq n$ ) other than  $i$ , let  $\alpha_{kj}$  ( $0 \leq j \leq r_k$ ) and  $\beta_{kj}$  ( $1 \leq j \leq r_k$ ) denote arcs of  $D_i$  considering  $D_i \cap D_k$  to be a part of  $D_k$ . We let  $a_{kj}$  and  $b_{kj}$  be the elements in  $RQ(T_i)$  assigned to  $\alpha_{kj}$  and  $\beta_{kj}$  respectively. Then we of course have relations  $a_{kj} = a_{k,j-1} *^{\varepsilon_{kj}} b_{kj}$  in  $RQ(T_i)$ . We remark that  $\alpha_{k0}$  and  $\alpha_{kr_k}$  are the same arc, and thus  $a_{k0}$  and  $a_{kr_k}$  are the same element in  $RQ(T_i)$ .

We inductively define a map  $f_i : \{a_{lj} \mid 1 \leq l \leq n, 0 \leq j \leq r_l\} \rightarrow RQ(T'_i)$ , distinguishing the elements  $a_{l0}$  and  $a_{lr_0}$ , as follows. To start with, we let  $f_i(a_{l0}) = a'_{l0}$  for all  $l$ . At each crossing, we set  $f_i(a_{lj}) = f_i(a_{l,j-1}) *^{\varepsilon_{lj}} f(p_i(b_{lj}))$ . Then, by induction, we have  $p_i(f_i(a_{lj})) = f(p_i(a_{lj}))$  and so  $f_i(a_{lj}) = f_i(a_{l,j-1}) *^{\varepsilon_{lj}} f_i(b_{lj})$  for all  $l$  and  $j$ . For each  $k$  ( $1 \leq k \leq n$ ) other than  $i$ , since  $f : RQ(L) \rightarrow RQ(L')$  is an isomorphism sending both of  $p_i(a_{k0})$  and  $p_i(a_{kr_k})$  to  $p_i(a'_{k0})$ , we have  $f_i(a_{k0}) = f_i(a_{kr_k}) = a'_{k0}$ . Therefore,  $f_i$  uniquely extends to a homomorphism  $f_i : RQ(T_i) \rightarrow RQ(T'_i)$ .

For the isomorphism  $g := f^{-1} : RQ(L') \rightarrow RQ(L)$ , we also have a homomorphism  $g_i : RQ(T'_i) \rightarrow RQ(T_i)$ . By construction, we obviously have  $g_i \circ f_i(a_{l0}) = a_{l0}$  for all  $l$ . Since  $a_{10}, a_{20}, \dots, a_{n0}$  generate  $RQ(T_i)$ ,  $g_i \circ f_i$  should be the identity map. Similarly,  $f_i \circ g_i$  is the identity map. Thus  $f_i : RQ(T_i) \rightarrow RQ(T'_i)$  is an isomorphism sending  $a_{i0}$  to  $a'_{i0}$ .

By Lemma 4.4, we have a 2-cocycle  $\theta'_i \in \tilde{Z}_{Q,qt}^2(RQ(L); \langle l'_i \rangle)$  derived from a section  $s' : RQ(L') \rightarrow RQ(T'_i)$  sending  $p_i(a_{i0})$  to  $a_{i0}$ . By a straightforward calculus, we have

$$\begin{aligned} & s'(f(p_i(a_{ij}))) \\ &= \begin{cases} \theta'_i(f(p_i(a_{i,j-1})), f(p_i(b_{ij})))^{-1} \cdot \{s'(f(p_i(a_{i,j-1}))) * s'(f(p_i(b_{ij})))\} & \text{if } \varepsilon_{ij} = 1, \\ \theta'_i(f(p_i(a_{ij})), f(p_i(b_{ij}))) \cdot \{s'(f(p_i(a_{i,j-1}))) *^{-1} s'(f(p_i(b_{ij})))\} & \text{if } \varepsilon_{ij} = -1 \end{cases} \end{aligned}$$

for all  $j$  ( $1 \leq j \leq r_i$ ). Therefore, by definition, we have

$$\begin{aligned} f_i(a_{ir_i}) &= \langle [\theta'_i]^{qt} \mid f \mid [K_i]^{qt} \rangle \cdot s'(f(p_i(a_{ir}))) \\ &= \langle [\theta'_i]^{qt} \mid \text{id} \mid [K'_i]^{qt} \rangle \cdot s'(f(p_i(a_{ir}))) \\ &= l'_i \cdot s'(f(p_i(a_{ir}))). \end{aligned}$$

Since  $s'(f(p_i(a_{ir}))) = a_{i0}$  and  $l'_i \cdot a_{i0} = a_{ir'_i}$ , the isomorphism  $f_i$  sends  $a_{ir_i}$  to  $a'_{ir'_i}$ .  $\square$

Now, we prove Theorem 4.2.

*Proof of Theorem 4.2.* Since  $RQ(L)$  is isomorphic to  $RQ(L')$ , if the  $i$ -th component of  $L'$  is trivial up to link-homotopy, then the  $i$ -th component of  $L$  should be trivial up to link-homotopy. Thus, in this case,  $L$  and  $L'$  are obviously link-homotopic to each other by the assumption. In the following, we assume that the  $i$ -th component of  $L'$  is not trivial up to link-homotopy.

We may assume that  $L$  and  $L'$  are the closures of  $\sigma$  and  $\sigma'$  respectively. Let  $\tau'$  be the pure braid satisfying  $\sigma' = \sigma \cdot \tau'$ . We note that the pure braid obtained from  $\tau'$  by removing the  $i$ -th component is trivial by the assumption.

Let  $\mu'$  and  $\nu'$  be nooses of  $L'$ , depicted in the left-hand side of Figure 8, which intersect with the  $i$ -th component at the start and end points of  $\tau'$  respectively. Then, in the light of Theorem 4.6,  $\mu'$  and  $\nu'$  are related to each other by homotopy and the moves depicted in Figure 4. Indeed,  $\mu'$  and  $\nu'$  are representatives of  $f_i(a_{ir_i})$  and  $a'_{ir'_i}$  respectively. We let  $\widehat{\mu}'$  be a noose of  $L'$ , depicted in the left-hand side of Figure 8, obtained from  $\mu'$  by moving its disk to the end point of  $\tau'$  along the  $i$ -th component of  $\tau'$ . We note that  $\widehat{\mu}'$  is obviously homotopic to  $\mu'$ , and so  $\widehat{\mu}'$  and  $\nu'$  are related to each other by homotopy and the moves. Thus a part of the ‘rope’ of  $\widehat{\mu}'$  (the image of the segment  $\{z \in \mathbb{C} \mid 1 \leq z \leq 5\}$ ) parallel to the  $i$ -th component of  $\tau'$  can be pulled out from  $L'$  by homotopy and the moves as depicted in the right-hand side of Figure 8. We claim that this deformation can be performed with the  $i$ -th component of  $\tau'$  keeping parallelness by link-homotopy.

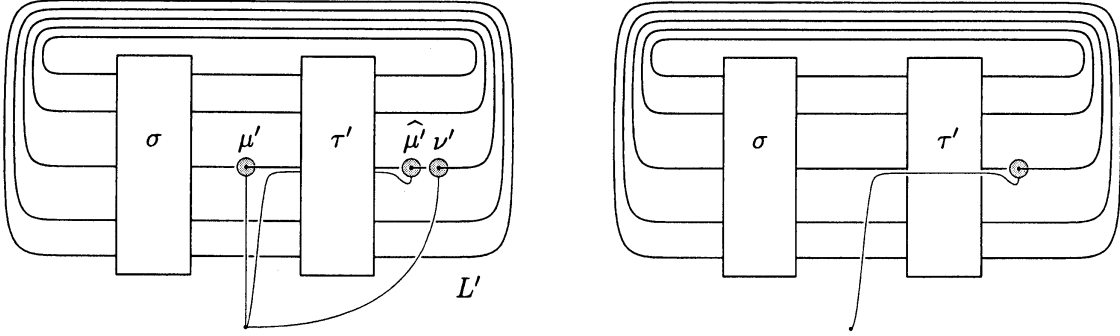


Figure 8:

Let  $\gamma'$  denote the part of the rope of  $\widehat{\mu}'$  parallel to the  $i$ -th component of  $\tau'$  and  $\Gamma'$  the  $i$ -th component of  $\tau'$  for simplicity. Since a crossing change for  $\gamma'$  can be performed with  $\Gamma'$  by a self-crossing change and type I moves as depicted in Figure 9, homotopy for  $\gamma'$  can be performed with  $\Gamma'$ . Obviously, a type I move for  $\gamma'$  can be performed with  $\Gamma'$  by a self-crossing change (and a crossing change for  $\gamma$  and a type I move if necessary, see Figure 10). We note that the self-crossing changes and the homotopy for  $\gamma'$  depicted in Figure 11 has an effect similar to a type II move. Hence the claim is true.

Pulling out  $\Gamma'$  from  $L'$  by link-homotopy keeps the parts of  $L'$  other than  $\Gamma'$  in appearance (see Figure 12). Thus the result is the closure of  $\sigma$ . It means  $L$  and  $L'$  are link-homotopic to each other.  $\square$

**Remark 4.7.** We can always pull out  $\Gamma'$  from  $L'$  by link-homotopy even though the pure

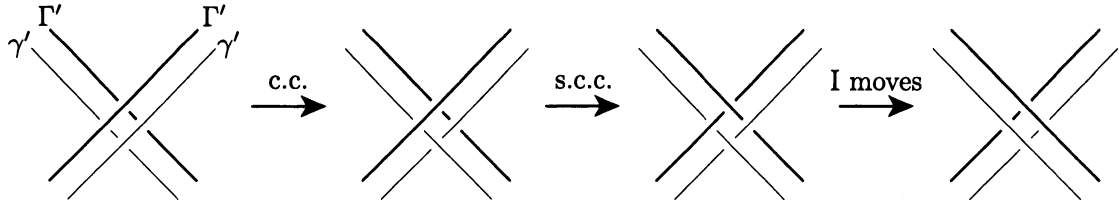


Figure 9:

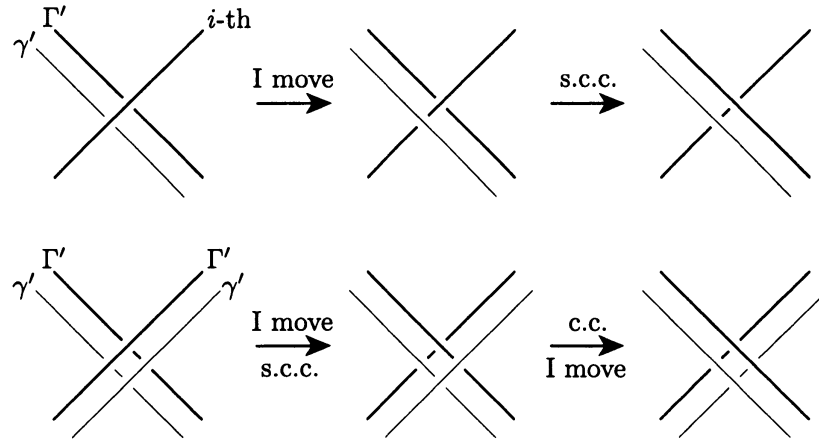


Figure 10:

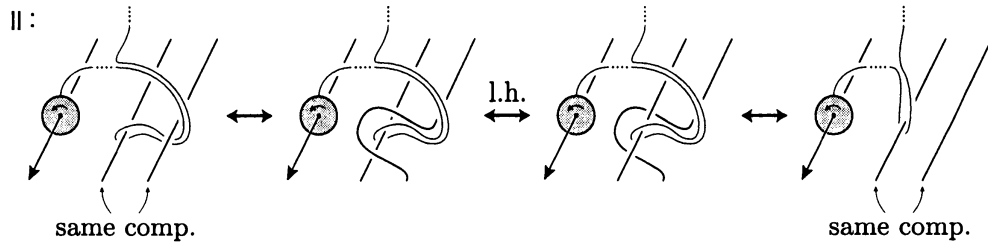


Figure 11:

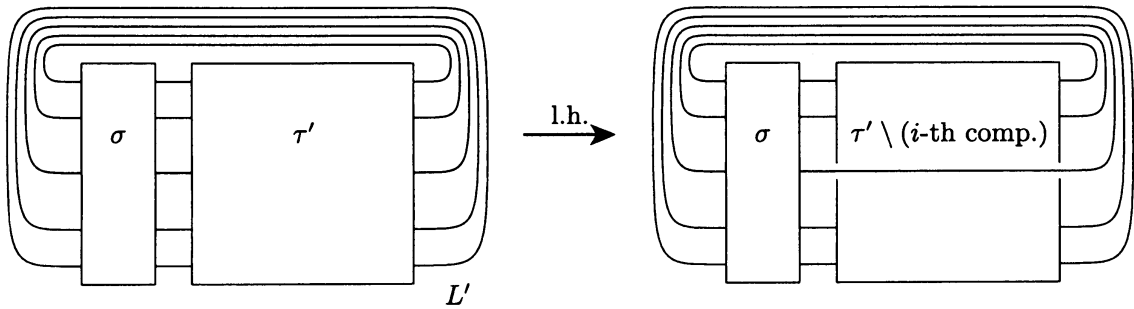


Figure 12:

braid obtained from  $\tau$  by removing the  $i$ -th component is not trivial. On the other hand, we can not always pull out  $\Gamma'$  from  $\sigma'$  by link-homotopy. This difference poses considerable difficulties in trying Conjecture 4.1 in this way.

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